Internship Report – M2 MPRI

Mnemetic Monads and Compactness

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Contents

This summary should be at most one (two-sided) page.

General context

We work in the setting of 2-category theory to understand equivalences given by taking *prime* or *compact* elements appearing in denotational semantics and categorical logic. These appear under the name of compact/finite/finitely presented object, atomic object, join irreducible, join prime. We will show that these can all be put into a common picture using the language of lax-idempotent (KZ) monads [Koc95; Zöb76], which are commonly thought of as *free cocompletions* under a class of colimits. In previous work these ideas have been considered in isolation. A first step towards this can be seen in [DL18] where the authors recover duality theorems by axiomatizing accessibility.

Research problem

This internship was aimed at introducing the study of *mnemetic lax-idempotent monads*, which provide a particularly pleasant contexts for studying cocompletions. The specific question was to find out when the inclusion from the basic data to the continuous algebras of the monad is a local equivalence. This question naturally set the stage for the general study of what a generator is with respect to a lax-idempotent monad.

This is a first step towards a general theory of generators for lax-idempotent monads, abstracting and unifying several constructions previously treated separately, such as in domain theory, realizability, and categorical logic. In a similar vein to [ADL23; DL18], we develop formal category theory in the setting of lax-idempotent pseudomonads.

Using this approach we obtain a synthetic approach to studying generators, that is without regard to the base 2-category. The main application so far is to recover results showing that certain objects are freely generated via different cocompletions. We also recover new notions of generator in the setting of free opfibrations on a functor [Koc13].

Contribution

We generalized the ideas of compact element, way below relation and having enough compact elements from domain Theory to the setting of lax-idempotent monads. These constructions are new and moreover give a conceptual answer to the original questions, namely taking generators is right adjoint to taking free continuous algebras. The advantage of the gained generality is that we can now capture many standard examples using the same technology while being able to formally prove theorems about them. The approach was to have a collection of examples from a wide range of topics from domain theory and topos theory to fibrations and realizability. This way we were able to find the common denominator of the constructions.

Arguments supporting its validity

The work has been mathematical in nature, thus all the claims come with proofs or references to their proofs. We claim that the given definitions are appropriate because of:

1. the generality and simplicity of the proofs they give rise to
2. the wide range of applications.

These applications range from lattice theory all the way to topos theory, type theory and higher category theory.

Our results rely on the observation that in CAT and in Pos, Kan extensions are computed pointwise, this assumption breaks down in a lot of other 2-Categories. Nonetheless the examples of CAT and Pos contain all the classical results we wanted to generalize and we plan on making the same treatment work for other 2-Categories by considering Virtual Equipments [CS09].

Summary and future work

We used the technology of lax-idempotent pseudomonads on 2-categories to define what it means to be a generator for an algebra using purely arrow theoretic arguments. We call this notion T-compactness, that is a notion of generator indexed by a given lax-idempotent pseudomonad. In the case of the Ind-construction, T-compactness recovers standard notion of compact/finitely presented object. Using this we developed the tools to study new notions of generator using a simple recipe.

More surprisingly we found that our treatment of lax-idempotent pseudomonads is a kind of lax-idempotent generalization of the work on monadic descent by [Mes06].

The main contribution is theorem 5.15, stating that taking generators is right adjoint to taking free continuous algebras. This gives a conceptual way of understanding equivalences between cauchy completions and T-compactly generated objects.

The next step will be to work in the more appropriate setting of virtual equipments to capture more exotic examples of generator. We have identified two such examples that we expect to yield new theorems with applications to semantics of dependent type theory, realizability and higher category theory. Namely the Monoidal Categories are lax-idempotent over Multicategories [Her01], and Opfibrations are lax-idempotent over CAT/B [Koc13].

Acknowledgement

I would like to thank Jonas Frey and Morgan Rogers and Damiano Mazza for always taking their time to discuss questions and ideas. I would like to thank the Locale Group in general for welcoming me so warmly and letting me present my work at their working group. I would also like to thank Julio Sampietro Christ, Nathanael Arkor, Joshua Wrigley, Vincent Moreau, Daniela Petrisan, Alexander Kurz, and Thomas Blom for the insightful discussions we had related to this work.

1 Introduction

A *cocompletion* of a category is a universal way of freely adjoining certain colimits. Under mild size conditions each of these cocompletions give rise to a pseudomonad on locally small categories. These satisfy a certain adjointness property which makes them more pleasant to work with, these are known as *lax-idempotent (or KZ) pseudomonads* [Koc95; Zöb76]. This lax-idempotent pseudomonad makes sense for other 2-categories than just locally small categories, for example partially ordered sets admit many naturally occurring lax-idempotent pseudomonads. Among them is the *sup-cocompletion* adjoining arbitrary joins to a poset, this is done using the *down-set construction.* The goal of this report is to develop a general framework for understanding how cocompletions give rise to "generators" through their associated lax-idempotent pseudomonads. For example in the case of the *directed-join-cocompletion* on posets, we obtain the standard notions of *finite element* and *way below relation* from domain theory.

2 Background

2.1 2-Categories

A 2-category is morally a category with arrows between the arrows, the prime example being CAT which has objects locally small categories, morphisms functors and 2-cells natural transformations.

Definition 2.1. A **2-category** $\mathcal{B}$ consists of:

- a class of objects (denoted A, B, C),
- for each pair A, B, a category $\mathcal{B}(A, B)$, whose objects are called **arrows**, and morphisms **2-cells**. We the composition of 2-cells is called **vertical composition**,
- for each object A an **identity arrow** $\mathrm{id}_A \in \mathcal{B}(A, A)$,
- for each triple of objects A, B, C a **horizontal composition** functor $(- \circ_{A,B,C} -) : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$,

making the following diagrams commute:

1. Associativity:

$$\begin{array}{ccc} \mathcal{B}(C,D) \times \mathcal{B}(B,C) \times \mathcal{B}(A,B) & \xrightarrow{\mathrm{id}_{\mathcal{B}(C,D)} \times \circ_{A,B,C}} & \mathcal{B}(C,D) \times \mathcal{B}(A,C) \\ {\scriptstyle \circ_{B,C,D} \times \mathrm{id}_{\mathcal{B}(A,B)}} \downarrow & & \downarrow {\scriptstyle \circ_{A,C,D}} \\ \mathcal{B}(B,D) \times \mathcal{B}(A,B) & \xrightarrow[\circ_{A,B,D}]{} & \mathcal{B}(A,D) \end{array}$$

2. Unit Laws:

$$\begin{array}{ccc} \mathcal{B}(A,B) & \xrightarrow{(\mathrm{id}_{\mathcal{B}(A,B)}, \mathrm{id}_A)} & \mathcal{B}(A,B) \times \mathcal{B}(A,A) \\ {\scriptstyle (\mathrm{id}_B, \mathrm{id}_{\mathcal{B}(A,B)})} \downarrow & \searrow^{\mathrm{id}_{\mathcal{B}(A,B)}} & \downarrow {\scriptstyle \circ_{A,A,B}} \\ \mathcal{B}(B,B) \times \mathcal{B}(A,B) & \xrightarrow{\circ_{A,B,B}} & \mathcal{B}(A,B) \end{array}$$

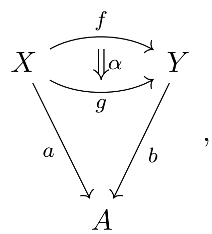
We will always write $f \circ g$ instead of $f \circ_{A,B,C} g$. For 2-cells we write vertical composition as $\alpha \circ \beta$ and horizontal composition as $\alpha \star \beta$. We also use the standard notation for whiskering $\alpha f := \alpha \star \mathrm{id}_f$ and $f\alpha := \mathrm{id}_f \star \alpha$ for an arrow f and a 2-cell α which are compatible. Horizontal composition has lower priority than vertical composition, i.e. is short for

$$\alpha \star \beta \circ \gamma \star \delta = (\alpha \star \beta) \circ (\gamma \star \delta).$$

Example 2.2. The 2-category Pos has objects partially ordered sets $(X, \leqslant)$ and hom-categories $Pos(X, Y)$ non-decreasing functions with their pointwise order viewed as a category.

Example 2.3. The 2-category CAT has as objects locally small categories and functors as arrows and natural transformations as 2-cells. Similarly 2-category Lex has as objects finitely complete locally small categories and finite limit preserving functors as arrows and natural transformations as 2-cells.

Example 2.4. For any 2-category $\mathcal{B}$ given an object $A \in \mathcal{B}_0$, we can construct the **slice-2-category** $\mathcal{B}/A$ which has objects arrows with codomain A, morphisms commuting triangles and 2-cells are of the form



,

i.e. 2-cells in $\mathcal{B}$ such that $b\alpha = a$ for $a, b \in \mathcal{B}/A$.

2.2 Universal Constructions in a 2-Category

Definition 2.5. An **adjunction** in a 2-category $\mathcal{B}$ is given by a pair of arrows $l : A \to B$ and $r : B \to A$ together with 2-cells $\eta : \mathrm{id}_A \Rightarrow rl$ and $\epsilon : lr \Rightarrow \mathrm{id}_B$ such that

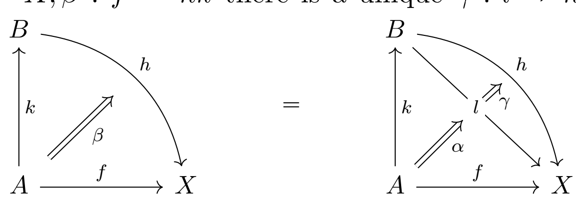
$$\epsilon l \circ l\eta = \mathrm{id}_l$$

and

$$r\epsilon \circ \eta r = \mathrm{id}_r.$$

We denote this situation by $l \dashv r$.

Definition 2.6. [SW78] Given arrows $f : A \to X$ and $k : A \to B$ in a 2-category $\mathcal{B}$, a **left extension of** f **along** k is a universal pair $(l : B \to X, \alpha : f \Rightarrow lk)$: That is for any other pair $h : B \to X, \beta : f \to hk$ there is a unique $\gamma : l \Rightarrow h$ such that $\beta = \gamma k \circ \alpha$.

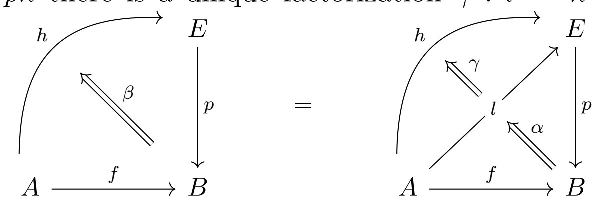


We write $\mathsf{lan}_k f$ for the arrow of the left extension of f along p when it exists.

A left extension (l, α) of f along k is **absolute** if given any arrow $g : X \to Y$, $(gl, g\alpha)$ is a left extension of gf along k.

The following are the lesser known duals of extensions, a good reference is [Di +17].

Definition 2.7. [SW78] Given arrows $f : A \to B$ and $p : E \to B$ in a 2-category $\mathcal{B}$, a **left lift of** f **along** p is a universal pair $(l : A \to E, \alpha : f \Rightarrow pl)$: That is for any other pair $h : A \to E, \beta : f \to ph$ there is a unique factorization $\gamma : l \Rightarrow h$ such that $\beta = p\gamma \circ \alpha$.



We write $\mathsf{lift}_p f$ for the arrow of the left lift of f along p when it exists.

A left lift (l, α) of f along p is **absolute** if given any arrow $g : C \to A$, $(lg, \alpha g)$ is a left lift of fg along p.

The definition is exactly saying that we have a left extension in the opposite 2-category, thus left lifts are unique up to unique iso. Of course we can dualize all these to obtain right extensions and right lifts in the same way.

The following lemma states that adjunctions are a special case of left lifts.

Lemma 2.8. *(2.4 of [Di +17]) Given arrows $l : A \to B$ and $r : B \to A$ such that $l \dashv r$ in $\mathcal{B}$ with unit $\eta : id_A \Rightarrow rl$, then (l, η) is an absolute left lift of r along id_A.*
Conversely if (η, l) is an absolute left lift of r along id_A, then we can construct a counit $\epsilon : lr \Rightarrow id_B$ such that (l, r, η, ϵ) is an adjunction.

We will make extensive use of inverters which should be thought of as a 2-categorical analogue of an equalizer which can take 2-cells into account. A more thorough treatment of inverters can be found in Section B1.15 of [Joh02].

Definition 2.9. In a 2-category $\mathcal{B}$ with arrows $f, g : A \to B$ and a 2-cell $\alpha : f \Rightarrow g$, an **inverter** of α is an arrow $\iota : X \to A$ such that:

- $\alpha\iota$ is invertible.
- For every object $Y \in \mathcal{B}$, the functor $- \circ \iota : \mathcal{B}(Y, X) \to \mathcal{B}(Y, A)$ is full and faithful with its replete image in arrows which invert α, that is if αf is invertible for some $f : Y \to A$, then there is some $\tilde{f} : X \to A$ such that $\iota \circ \tilde{f} \cong f$.

3 Basics of Lax-Idempotent Monads

3.1 Lax-Idempotent Monads as a Setting for Cocompletions

Lax-idempotent pseudomonads capture the idea of *cocompletion*. The most common examples are the down-set completion and the ideal completion on posets and the presheaf construction or the Ind-completion on locally small categories. The organizing power of lax-idempotent monads is for example being used by [Blo25] to generalize common 1-categorical arguments to that of $(\infty, 1)$-categories in a painless way.

It can also be seen from [ADL23] that lax-idempotent monads can give a satisfactory context for formal category theory. Moreover we will see that certain dualities can also be viewed

from this perspective.
A definition of pseudomonads can be found in Appendix A.

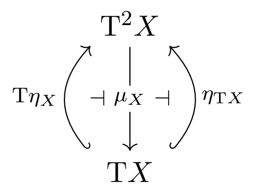
Definition 3.1. A pseudomonad $(\mathrm{T} : \mathcal{B} \to \mathcal{B}, \mu, \eta)$ on a 2-category $\mathcal{B}$ is lax-idempotent (or KZ) [Koc95; Zöb76] when for every object $X \in \mathcal{B}$, we have an adjunction

$$\mu_X \dashv \eta_{\mathrm{T}X},$$

such that the invertible counit is given by the left unitor $l_X : \mu_X \circ \eta_{\mathrm{T}X} \Rightarrow \mathrm{id}_{\mathrm{T}X}$ In this case it was shown (cf. [Joh02]), we also get

$$\mathrm{T}\eta_X \dashv \mu_X.$$

This situation can then be summarized by saying for each object X, we obtain two adjunctions



We will refer to lax-idempotent pseudomonads as **lax-idempotent monads**.

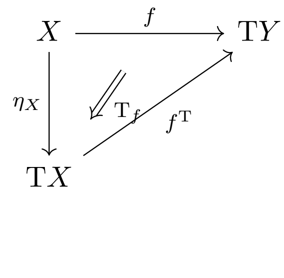
In [MW12], the authors showed that a lax-idempotent monad can be specified very efficiently.

Definition 3.2. A **left Kan pseudomonad** T on a 2-category $\mathcal{B}$ is given by

– a function $\mathrm{T}_0 : \mathcal{B}_0 \to \mathcal{B}_0$

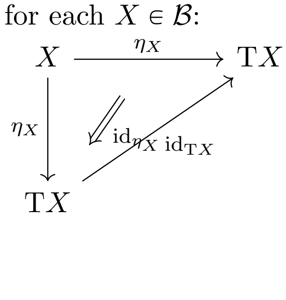
– for each object $X \in \mathcal{B}$, an arrow $\eta_X : X \to \mathrm{T}X$

– for every arrow $f : X \to \mathrm{T}Y$ a left extension along η_X:

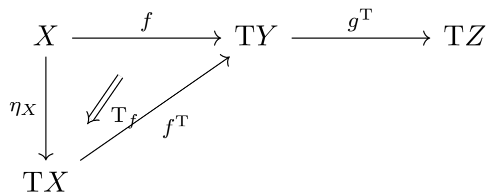


satisfying that

1. the following is a left extension for each $X \in \mathcal{B}$:



2. for each $f : X \to \mathrm{T}Y$ and $g : Y \to \mathrm{T}Z$:



, the arrow $g^{\mathrm{T}} \circ f^{\mathrm{T}}$ is a left extension of $g^{\mathrm{T}} \circ f$ along η_X.

Lemma 3.3. *[MW12] Every lax-idempotent monad* $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ *arises from a left Kan pseudomonad with the action on arrows given by*

$$\mathrm{T}(f : A \to B) := \mathsf{lan}_{\eta_A}(\eta_B f)$$

and the multiplication is given by

$$\mu_X := \mathsf{lan}_{\eta_{\mathrm{T}X}} id_{\mathrm{T}X}.$$

3.2 Many Examples of Lax Idempotent Monads

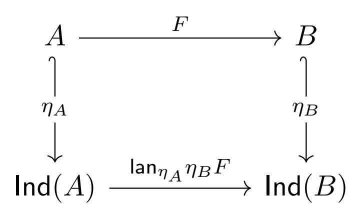
The following example is the general shape that we will consider here.

Proposition 3.4. *There is a lax-idempotent monad* $\mathsf{Ind} : \mathsf{CAT} \to \mathsf{CAT}$ *specified by the following left Kan pseudomonad on* CAT*:*

- *On objects, define* $\mathsf{Ind}(C) \subseteq \mathsf{Set}^{C^{op}}$ *defined on the presheaves which are small filtered colimits of representables*

- *The unit maps* $\eta_C : C \to \mathsf{Ind}(C)$ *are given by the Yoneda embedding restricted on its codomain*

Section 3.1 shows that the action on morphisms (functors) $F : A \to B$ *is given by the following left Kan extension:*



Moreover the multiplication is given on objects $C \in \mathsf{CAT}$ *by*

$$\mu_C(\psi \in \mathsf{Ind}^2(C)) := \mathsf{colim}(\int \psi \to \mathsf{Ind}(C)).$$

Here $\int \psi$ *is the category of elements of* ψ *[Awo06].*

Example 3.5. [PT21] There is a lax-idempotent monad $\mathcal{P} : \mathsf{CAT} \to \mathsf{CAT}$ specified by the following left Kan pseudomonad on CAT:

- On objects, define $\mathcal{P}(C) \subseteq \mathsf{Set}^{C^{op}}$ defined on the presheaves which are small colimits of representables

– The unit maps $\eta_C : C \to \mathcal{P}(C)$ are given by the Yoneda embedding on its codomain

Example 3.6. [Gie+80] A subset S of a poset X is **down-closed** if $s \in S$ and $t \leqslant s$, implies $t \in S$. A subset S of a poset X is **directed** if it is non-empty and for any $s, t \in S$ there is $u \in S$ such that $s \leqslant u$ and $t \leqslant u$.

There is a lax-idempotent monad $\mathsf{Idl} : \mathsf{Pos} \to \mathsf{Pos}$ specified by the following left Kan pseudomonad on Pos:

– On objects, define $\mathsf{Idl}(X) := \{S \subseteq X | \text{directed and down-closed}\}$

– The unit maps $\eta_X : X \to \mathsf{Idl}(X)$ are given by principal down-sets $\eta_X(x) := y \in X | y \leqslant x$.

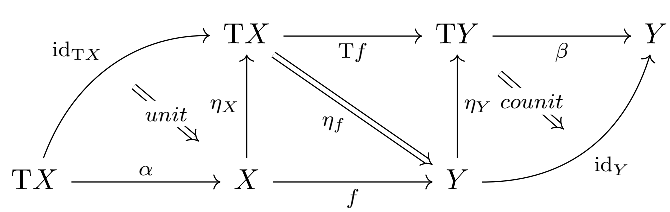
3.3 The 2-category of Algebras for a Lax Idempotent Monad

For pseudomonads there are multiple notions of algebra, we will only consider the pseudo-algebras (cf. Section A) and content ourselves with the fact that for lax-idempotent monads they are easy to detect. When it is clear from context we will refer to pseudo-T-algebras for a given lax-idempotent monad T as **algebras**.

Proposition 3.7. *[Koc95; Zöb76] The algebras of a lax-idempotent monad* $(\mathrm{T} : \mathcal{B} \to \mathcal{B}, \mu, \eta)$ *are objects* $X \in \mathcal{B}$ *such that the unit* η_X *has a left adjoint with invertible counit* ϵ*, that is* $\alpha \dashv \eta_X$ *and* $\epsilon : \alpha\eta_X \cong id$*.*

Remark 3.8. This means that being an algebra for a lax-idempotent monad is a **property**.

Remark 3.9. Let $(X, \alpha), (Y, \beta)$ be algebras for a lax-idempotent monad $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ and let $f : X \to Y$ be an arrow $\mathcal{B}$. We can always construct a 2-cell $\tilde{f} : \beta \mathrm{T} f \Rightarrow f\alpha$ as follows:



This is exactly the **mate** [KS06] of η_f.

Example 3.10. In the case of $\mathsf{Idl} : \mathsf{Pos} \to \mathsf{Pos}$, take $X, Y \in \mathsf{Dcpo}$ and consider a non-decreasing function $f : X \to Y$, the 2-cell $\tilde{f}$ tells us that for any directed set $S \subset X$, $\bigvee f(S) \leqslant f \bigvee S$.

Definition 3.11. [KL97] A morphism $f : (X, \alpha) \to (Y, \beta)$ of algebras for a lax-idempotent monad is an arrow $f : A \to B$ such that the mate $\tilde{f}$ is invertible.

Example 3.12. For any morphism $f : X \to Y$, $\mathrm{T}f : \mathrm{T}X \to \mathrm{T}Y$ is an algebra morphism.

Proposition 3.13. *[KL97] For any lax-idempotent monad* $\mathrm{T} : \mathcal{B} \to \mathcal{B}$*, there is a* 2*-category* T-Alg *which has as objects algebras, morphisms algebra morphisms and as* 2*-cells the original* 2*-cells of* $\mathcal{B}$*.*

Example 3.14. The monad $\mathcal{P}$ from 3.5 has as algebras locally small cocomplete categories $\mathsf{CoCompCat}$.
The monad Ind from 3.6 has as algebras posets admitting all directed joins, these are called **directed complete partialy ordered sets (DCPO)**, we denote this 2-category of algebras by Dcpo.

The following lemma is a formal version of the fact that left adjoints preserve colimits.

Proposition 3.15. *For lax-idempotent monads, left adjoints are always morphisms of algebras.*

Proof. Consider algebras (X, α) and (Y, β) for a lax-idempotent monad $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ and assume we have a morphism $l : X \to Y \in \mathcal{B}$ with a right adjoint $r : Y \to X$. We wish to show that $l\alpha \cong \beta \mathrm{T} l$ via a canonical isomorphism. To do this notice that $\alpha \dashv \eta_X$ and $l \dashv r$. Now since adjoints compose, we have that

$$l\alpha \dashv \eta_X r$$

And we have that $\eta_X r \cong \mathrm{T} r \eta_Y$ by naturality, giving us that

$$\beta \mathrm{T} l \dashv \mathrm{T} r \eta_Y.$$

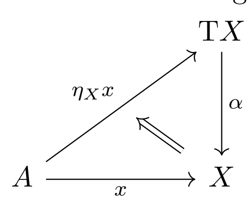
Now left adjoints are unique up to unique isomorphism, so $l\alpha \cong \beta \mathrm{T} l$. □

Corollary 3.16. *For any algebra (X, α), $\alpha : \mathrm{T}X \to X$ is an algebra morphism.*

4 Compactness

Now we introduce a notion of compactness parametrized by a lax-idempotent monad. The idea is to recover the usual notions of generator appearing in the literature, for example projective objects, finitely presented objects or connected objects and unify their study.

Definition 4.1. A generalized element $x : A \to X$ of an algebra (X, α) is **T-compact** if $(\eta_X x, c_X^{-1} : x \Rightarrow \alpha \eta_X x)$ is an absolute left lift of x along α



where $c_X : \alpha \eta_X \Rightarrow \mathrm{id}_X$ is the invertible counit of $\alpha \dashv \eta_X$. In particular this means that we have a natural bijection we have a natural bijection:

$$\frac{\eta_X x \Rightarrow u}{x \Rightarrow \alpha u}$$

stable under precomposition.

Remark 4.2. The way to think about this is to say that around x, α behaves like a right adjoint to η_X.

Proposition 4.3. *For the* Ind*-completion, a finitely presented object* $X \in \mathcal{A}$[1] *is exactly one for which* $X : 1 \to A$ *is* Ind*-compact.*

Proof. Assume $X : 1 \to \mathcal{A}$ is Ind-compact and consider a diagram $D : I \to \mathcal{A}$ with I a filtered category. Denote $\psi := \mathsf{colim}(\eta_{\mathcal{A}} \circ D)$ the associated object in $\mathsf{Ind}(\mathcal{A})$. So now:

$$\begin{aligned}\mathcal{A}(X, \mathsf{colim} D) &= \mathcal{A}(X, \alpha\psi) \\ &\cong \mathsf{Ind}(\mathcal{A})(\eta_{\mathcal{A}} X, \psi) \\ &\cong \mathsf{Ind}(\mathcal{A})(\eta_{\mathcal{A}} X, \mathsf{colim} \int \psi \to \mathcal{A} \to \mathsf{Ind}(\mathcal{A})) \\ &\cong \mathsf{colim}_{i \in I} \mathsf{Ind}(\mathcal{A})(\eta_{\mathcal{A}} X, \eta_{\mathcal{A}} C_i) \\ &\cong \mathsf{colim}_{i \in I} \mathcal{A}(X, C_i)\end{aligned}$$

The second to last step used that all representables are finitely presented. The converse direction is easy. □

Remark 4.4. This style of proof crucially relies on the fact that $\eta_{\mathcal{A}} : \mathcal{A} \to \mathsf{Ind}(\mathcal{A})$ is fully faithful. This suggests that this style of argument fits into the setting of **Yoneda structures** [Wal17].

Example 4.5. For the monad Idl from theorem 3.6, given a $X \in$ Dcpo an element $x : 1 \to X$ is Idl-compact exactly when it is **compact or finite** in the sense of [Gie+80].

Remark 4.6. For the complete story to work one would need to work in the general setting of a virtual equipment to recover the right notion of generator, we will leave this for future work.

The following definition is a generalization of coherent morphisms of coherent locales [Joh82].

Definition 4.7. An algebra morphism $f : (X, \alpha) \to (Y, \beta)$ is **coherent** if it preserves compact objects, that is if $x : A \to X$ is compact then $fx : A \to Y$ is compact.

Proposition 4.8. *Given algebras* (X, α) *and* (Y, β)*, any left adjoint* $l : (X, \alpha) \to (Y, \beta)$ *in the* 2*-category of algebras* T-Alg *is coherent.*

Proof. Let $r : (Y, \beta) \to (X, \alpha)$ be an algebra morphism which is right adjoint to l in T-Alg. So it is in particular a right adjoint in $\mathcal{B}$.

We have to show that given a compact object $x : A \to X$ of (X, α), we have that $lx : A \to Y$ is compact in (Y, α). So take any diagram $u : A \to \mathrm{T}Y$ in Y and notice

[1] Recall that an object $X \in \mathcal{A}$ is finitely presented if for any filtered diagram $D : I \to \mathcal{A}$

$$\mathcal{A}(X, \mathsf{colim}_{i \in I} D(i)) \cong \mathsf{colim}_{i \in I} \mathcal{A}(X, D(i))$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\eta_Y lx \Rightarrow u}{\mathrm{T}l\eta_X x \Rightarrow u}}{\eta_X x \Rightarrow \mathrm{T}ru}}{x \Rightarrow \alpha \mathrm{T}ru}}{x \Rightarrow r\beta u}}{lx \Rightarrow \beta u}$$

Thus lx is compact. □

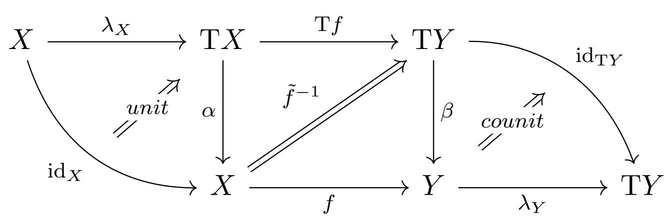
5 Factorization

Next we introduce continuous algebras for lax-idempotent monads on a 2-category and describe a way of deducing from every lax-idempotent monad on a 2-category with enough limits an adjunction between the base category and the category of continuous algebras. A similar idea has been studied before in the descent theory for categories of modules by [Mes06], in which the conceptual meaning of the induced adjoint was not explored. We will show that the right adjoint from continuous algebras to the base 2-category takes a continuous algebras to its object of generators.

For the rest of this section we consider a fixed 2-category $\mathcal{B}$ together with a lax-idempotent monad $(\mathrm{T} : \mathcal{B} \to \mathcal{B}, \mu, \eta)$.

5.1 Continuous Algebras of Lax-Idempotent Monads

Definition 5.1. A **continuous algebra** for T is a triple (X, α, λ_X) such that (X, α) is a T-algebra and α has a further left adjoint $\lambda_X : X \to \mathrm{T}X$. A **morphism of continuous algebras** is a morphism of algebras $f : (X, \alpha, \lambda_X) \to (Y, \beta, \lambda_Y)$ such that the mate $\hat{f} : \mathrm{T}f\lambda_X \cong \lambda_Y f$ of $\tilde{f}$, which we defined in theorem 3.9 is invertible



Notice that the triple of adjoints for a continuous algebra (X, α, λ_X) always induces a 2-cell, which we will denote $\theta_X : \lambda_X \Rightarrow \eta_X$.

Example 5.2. For theorem 3.5, the continuous algebras are the completely distributive categories CDistCat from [MRW12]. For theorem 3.6, the continuous algebras are the continuous posets [Gie+80], hence the name.

Remark 5.3. Every free algebra of a lax-idempotent monad is continuous by definition.

The next lemma should be taken as a categorification of: an element is compact in the sense of domain theory if and only if it is way below itself, see for example chapter 1 of [Gie+80].

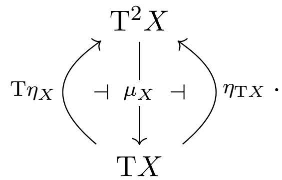
Lemma 5.4. *For a continuous algebra (X, α), a generalized element $x : A \to X$ is* T*-compact if and only if $\theta_X x$ is invertible.*

Proof. (⇒) Since we have that $\eta_X x$ and $\lambda_X x$ are both left lifts there is a canonical iso between them and it must invert θ_X.

(⇐) Since $\theta_X x$ is invertible, $\eta_X x \cong \lambda x$, and λ is a left adjoint and thus an absolute left lift of the identity, hence $\eta_X x$ is an absolute left lift. □

Corollary 5.5. *The unit $\eta_X : X \to \mathrm{T}X$ is always compact.*

Proof. By theorem 5.3, any $\mathrm{T}X$ is a continuous algebra with the adjoint cylinder given by



Notice that naturality tells us that $\eta_{\mathrm{T}X}\eta_X \cong \mathrm{T}\eta_X\eta_X$ and the fact that this isomorphism is given by $\theta_{\mathrm{T}X}$ can be found in [Koc95]. Thus the unit is compact. □

Lemma 5.6. *Coherent algebra morphisms between free algebras are automatically morphisms of continuous algebras.*

Proof. Take objects $X, Y \in \mathcal{B}$ and an algebra morphism $f : \mathrm{T}X \to \mathrm{T}Y$ such that for any $x : A \to X$ compact, $fx : A \to Y$ compact. We want to show that $\mathrm{T}f \circ \mathrm{T}\eta_X \cong \mathrm{T}\eta_Y f$. By theorem 5.5 the unit η_X is always compact. Thus $f\eta_X$ is compact by assumption. Thus by theorem 5.4:

$$\begin{aligned} \mathrm{T}f\mathrm{T}\eta_X\eta_X &\cong \mathrm{T}f\eta_{\mathrm{T}X}\eta_X \\ &\cong \eta_{\mathrm{T}Y}f\eta_X \\ &\cong \mathrm{T}\eta_Y f\eta_X \end{aligned}$$

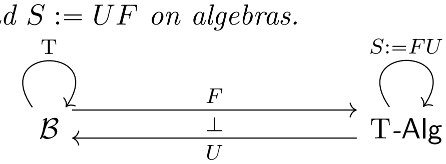
Now notice that all the morphisms involved here are in fact algebra morphisms so by the universal property of free algebras

$$\mathrm{T}f \circ \mathrm{T}\eta_X \cong \mathrm{T}\eta_Y f.$$

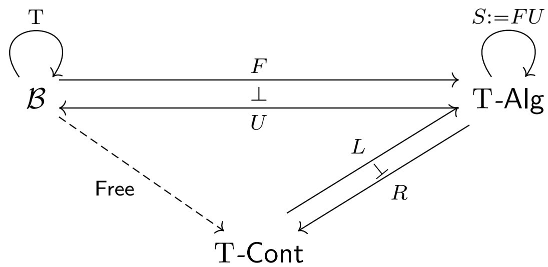
□

Remark 5.7. In light of theorem 5.6 one could hope that coherent algebra morphisms are automatically in the image of $Free$. This is not the case. We will see a counterexample in Section 7.

Lemma 5.8. *[Koc95] The category of continuous algebras* T-Cont *for* T *is exactly the category of coalgebras for the comonad $S := UF$ on algebras.*



Corollary 5.9. *There is a comparison functor* $\mathsf{Free} : \mathcal{B} \to \mathrm{T}$-$\mathsf{Cont}$ *induced by the adjunction $F \dashv U$.*



Free *is given on objects and morphisms by* T. *In particular this means that for any* $f : X \to Y$, $\mathrm{T}f : \mathrm{T}X \to \mathrm{T}Y$ *is a morphism of continuous algebras.*

Lemma 5.10. *Continuous algebra morphisms are coherent.*

Proof. Take a morphism of continuous algebras $f : (X, \alpha, \lambda_X) \to (Y, \beta, \lambda_Y)$ and a compact object $x : A \to X$ of X. And we would like to show that $fx : A \to Y$ is compact in Y:

$$\cfrac{\cfrac{\cfrac{\cfrac{\eta_Y f x \Rightarrow u}{\mathrm{T}f\eta_X x \Rightarrow u}}{\mathrm{T}f\lambda_X x \Rightarrow u}}{\lambda_Y f x \Rightarrow u}}{fx \Rightarrow \beta u}$$

Which concludes the proof. □

5.2 A Factorization Theorem

Definition 5.11. An **adjoint cylinder** in a bicategory $\mathcal{B}$ consists of two adjunctions $l \dashv a \dashv u$ such that $au \cong id$ and $al \cong id$. This configuration always induces a two-cell $\theta : l \Rightarrow u$.

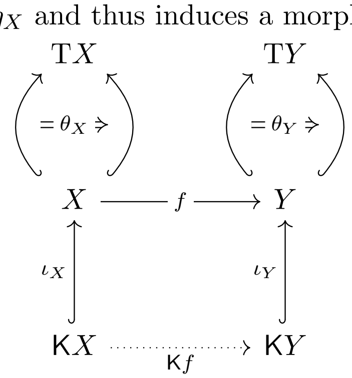
Example 5.12. Every continuous algebra (X, α, η_X) induces an adjoint cylinder $\lambda_X \dashv \alpha \dashv \eta_X$.

Definition 5.13. We say that a 2-category $\mathcal{B}$ has **inverters of adjoint cylinders** if the induced two cell of any adjoint cylinder admits an inverter.

This can be seen as a lax version of having coreflexive equalizers.

Proposition 5.14. *If* $\mathcal{B}$ *has inverters of adjoint cylinders, then there is a pseudofunctor* $\mathsf{K} : \mathsf{T\text{-}Cont} \to \mathcal{B}$ *taking a continuous algebra to the inverter of its induced adjoint cylinder (cf. theorem 5.12).*

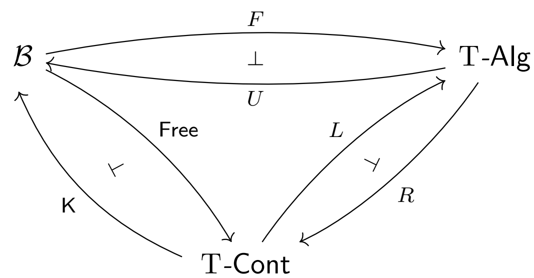
Proof. On objects $\mathsf{K}(X, \alpha, \lambda_X)$ is the inverter of θ_X. By Section 5.1, any continuous algebra morphism $f : (X, \alpha, \lambda_X) \to (Y, \beta, \lambda_Y)$ preserves compact objects, which by theorem 5.4 inverts the canonical 2-cell $\theta_X : \lambda_X \Rightarrow \eta_X$ and thus induces a morphism between the inverters.



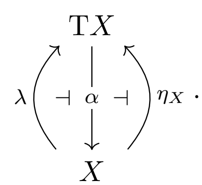
By Section 2.2 this assignment is functorial and that it extends to a pseudofunctor. □

Now we are ready to state the main theorem of this section.

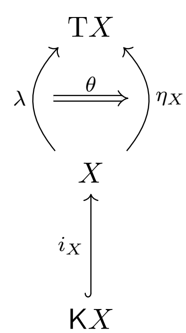
Theorem 5.15. *If $\mathcal{B}$ has inverters of adjoint cylinders and (T, μ, η) is a lax-idempotent monad on $\mathcal{B}$, then* $\mathsf{Free} \dashv \mathsf{K}$ *form a biadjunction between $\mathcal{B}$ and the category of continuous algebras* $\mathrm{T}\text{-}\mathsf{Cont}$ *making the left adjoints in the following diagram commute:*



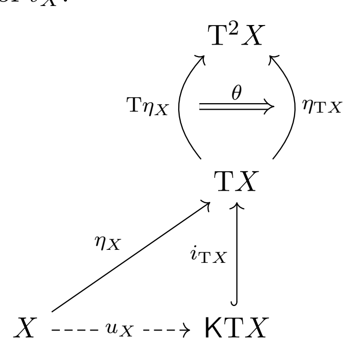
Proof. The existence of the pseudofunctor K is by theorem 5.14. Explicitly, given a continuous algebra



We can consider the inverter of the induced 2-cell $\theta : \lambda \Rightarrow \eta_X$:



Which we denote by $(\mathsf{K}X, \iota_X)$. The unit $u_X : X \to \mathsf{K}\mathrm{T}X$ of the adjunction $\mathsf{Free} \dashv \mathsf{K}$ is given by the universal property of ι_X.



The counit $c_X : \mathrm{T}\mathsf{K}X \to X$ is given by the following composite:

$$\mathrm{T}\mathsf{K}X \xrightarrow{\mathrm{T}i_X} \mathrm{T}X \xrightarrow{\alpha} X \ .$$

Notice that c_X is an algebra morphism.

The triangle identities will be essentially automatic once we show that c_X is indeed a morphism of continuous algebras.

Consider:

$$\begin{aligned} \mathrm{T}c_X \mathrm{T}\eta_{\mathsf{K}X} &\cong \mathrm{T}(\alpha \mathrm{T}\iota_X)\mathrm{T}\eta_{\mathsf{K}X} \\ &\cong \mathrm{T}(\alpha \mathrm{T}\iota_X \eta_{\mathsf{K}X}) \\ &\cong \mathrm{T}(\alpha \eta_X \iota_X) \\ &\cong \mathrm{T}(\iota_X) \end{aligned}$$

and on the other hand:

$$\lambda c_X \cong \lambda \alpha \mathrm{T} i_X$$

Now by Section 3.3, λ_X is an algebra morphism. Hence λc_X and $\mathrm{T}(\iota_X)$ are algebra morphisms defined on the free algebra $\mathrm{T}\mathsf{K}X$. So we can use the universal property of free algebras to check isomorphism on the generators by precomposing with $\eta_{\mathsf{K}X}$.

$$\begin{aligned} \lambda c_X \eta_{\mathsf{K}X} &\cong \lambda \alpha \mathrm{T} i_X \eta_{\mathsf{K}X} \\ &\cong \lambda \alpha \eta_X \iota_X \\ &\cong \lambda i_X \\ &\cong \eta_X i_X \\ &\cong \mathrm{T} i_X \eta_{\mathsf{K}X} \end{aligned}$$

Thus $\lambda c_X \cong \mathrm{T}c_X \mathrm{T}\eta_{\mathsf{K}X}$ making c_X is a continuous algebra morphism.

Therefore $\tilde{F} \dashv \mathsf{K}$. $\square$

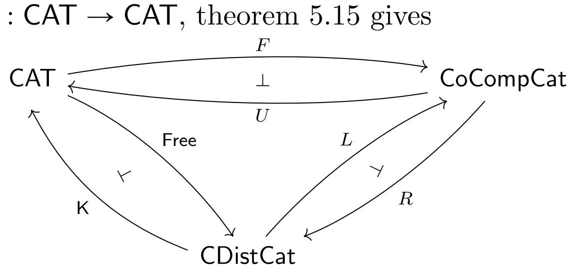
Corollary 5.16. *For any $X \in \mathcal{B}$ admitting a* T*-algebra structure, we have that* $\mathsf{K}\mathrm{T}X \cong X$.

Proof. Since $L\tilde{F} = F$ we have that $\mathsf{K}R \cong U$ by uniqueness of adjoints up to iso. Thus

$$\begin{aligned} \mathsf{K}R(X, \alpha) &\cong \mathsf{K}(\mathrm{T}X, \mu_X) \\ &\cong U(X, \alpha) \\ &= X \end{aligned}$$

$\square$

Example 5.17. For $\mathcal{P} : \mathsf{CAT} \to \mathsf{CAT}$, theorem 5.15 gives



The adjunction $\mathsf{Free} \dashv \mathsf{K}$ is idempotent and the induced monad on CAT is the usual Cauchy completion monad.

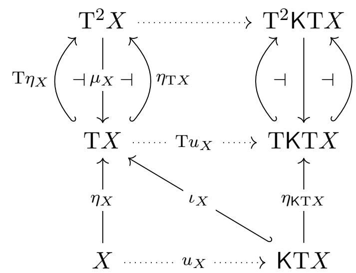
Definition 5.18. Going off the example $\mathcal{P}$, we call the induced monad $\mathsf{KFree} : \mathcal{B} \to \mathcal{B}$ the **T-Cauchy monad**.

We will see in Section 7 that this monad need not be idempotent.

5.3 Mnemetic Monads

In light of theorems 5.15 and 5.18 we would like to know when $\mathsf{Free} \dashv \mathsf{K}$ becomes idempotent.

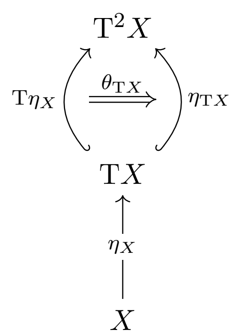
Definition 5.19. A lax-idempotent monad $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ on a 2-category is **pre-mnemetic** if the inverter $\iota_X : \mathsf{K}\mathrm{T}X \to \mathrm{T}X$ of $\theta_{\mathrm{T}X} : \mathrm{T}\eta_X \Rightarrow \eta_{\mathrm{T}X}$ exists and the arrow $\mathrm{T}u_X$ in the diagram



is an equivalence, where u_X is the canonical arrow into the inverter.

Remark 5.20. For pre-mnemetic monads the T-Cauchy monad is idempotent, in which case it truly is a kind of Cauchy completion.

Definition 5.21. A lax-idempotent monad $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ on a 2-category is **mnemetic** if η_X is an inverter of $\theta_{\mathrm{T}X}$:



Remark 5.22. Mnemetic monads are a lax-idempotent variation of the monads of **descent type** as defined in [Mes06].

It is now easy to check that each mnemetic monad is in fact pre-mnemetic. Moreover we get the following easy consequences.

Proposition 5.23. *A lax-idempotent monad* $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ *on a 2-category* $\mathcal{B}$ *with inverters of adjoint cylinders is pre-mnemetic if and only if the induced adjunction* $\mathsf{Free} \dashv \mathsf{K}$ *is idempotent. Moreover we call fixed points the* ***Cauchy complete objects*** *corresponding to* T *and the objects generated by their* T*-compact objects preserving the generators.*

Corollary 5.24. *A lax-idempotent monad* T *on a category with inverters of adjoint cylinders is mnemetic if and only if the induced functor* Free *is a local equivalence.*

Remark 5.25. This holds even without the inverters existing.

Theorem 5.26. *Any pre-mnemetic monad* T *factors as an idempotent monad followed by a mnemetic monad.*

Remark 5.27. Mnemetic monads are always locally fully faithful in the sense of [Wal17], thus mnemetic monads are particularly close to Yoneda structures.

Remark 5.28. This means that a lax-idempotent monad is mnemetic if the base 2-category has Cauchy complete objects.

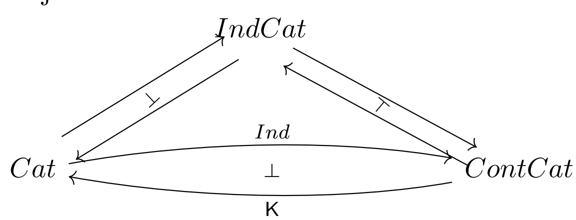
Definition 5.29. A continuous algebra (X, α) has **enough** T**-compact objects** if the counit $c_X : \mathsf{T}\mathsf{K}X \to X$ is an equivalence.

From this we recover a similar theorem to that of [Mes06] for **nuclear adjunctions** [PH21] (adjunctions which are monadic and comonadic).

Corollary 5.30. *The left adjoint* $F : \mathrm{T}\text{-}\mathsf{Alg} \to \mathcal{B}$ *is comonadic if and only if* T *is mnemetic and each continuous algebra has enough compact objects.*

We are due for some examples.

Example 5.31. If we instantiate theorem 5.15 with the *Ind* construction on locally small categories we obtain an adjunction:



Moreover since taking the compact elements of the ind-completion recover the Cauchy completion we have an induced equivalence, once we restricted to small categories

$$Cat_{small,cc} \simeq Acc_{fin},$$

where Acc_{fin} has finitely accessible categories with functors preserving compact objects.

Moreover if we do the same for the Ind-completion on finitely cocomplete categories we get a kind of covariant Gabriel Ulmer Duality:

$$Rex \simeq LFP.$$

A usecase of the mnemeticity condition is that detecting free subalgebras simplifies. The proof can be found in the appendix C.6.

Proposition 5.32. *If* $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ *is mnemetic and* $\mathcal{B}$ *has inverters of adjoint cylinders, then if* $A \in \mathcal{B}$ *and* (X, α, λ_X) *such that*

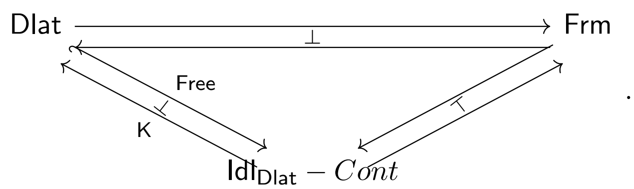
$$X \underset{l}{\overset{i}{\rightleftarrows}} \mathrm{T}A \quad (i \dashv l)$$

with $i : X \to \mathrm{T}A$ *a representably full and faithful continuous algebra morphism. In other words* X *is a reflective continuous subalgebra of* $\mathrm{T}A$*. Then* $c_X : \mathsf{T}\mathsf{K}X \to X$ *is an equivalence of continuous algebras.*

The following came out of conversations with Nathanael Arkor.

Corollary 5.33. *If* (X, α, λ) *is a continuous algebra such that* λ *has a further left adjoint, then* X *is a free algebra. Thus the adjunction* $Cofree - Forget$ *is nuclear.*

Example 5.34. We have that $\mathsf{Frm} \to \mathsf{Dlat}$ is monadic with the left adjoint given on objects by Idl. From this it is easy to see that the induced monad on Dlat is mnemetic. Thus theorems 5.15 and 5.24 apply



The continuous algebras are continuous locales with a stronger condition on the generators, since $\lambda : L \to \mathsf{Idl}_{\mathsf{Dlat}}(L)$ has to be a Dlat morphism. These are called **stably locally compact locales** [Tow22]

Note that compactness stays the same as for Idl on Pos.

Now the essential image of Free is exactly CohLoc^{op} from [Joh82] giving us the known equivalence $\mathsf{Dlat} \simeq \mathsf{CohLoc}^{op}$.

Remark 5.35. There ought to be a similar version for toposes of this.

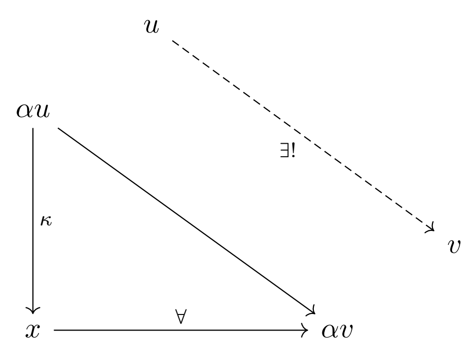
6 Way Below arrows and Slim Cocones

We now give a generic way of obtaining a way below relation from a lax-idempotent monad. Answering the folklorish question of how to generalize the wavy arrows appearing in the context of continuous categories.

Definition 6.1. Let $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ be lax-idempotent, and let (X, α) be an algebra.

For $u : A \to \mathrm{T}X$ and $x : A \to X$, we view 2-cells $\kappa : \alpha \circ u \to x$ as (parametrized) **cocones**. Since in ordinary category theory, cocones are in correspondence with arrows out of colimits.

We call the cocone κ **slim**, if for every $v : A \to \mathrm{T}X$ and $\phi : x \to \alpha \circ v$ there exists a unique $\gamma : u \to v$ with $\alpha * \gamma = \phi\kappa$:



Moreover this property should be **absolute** in the sense that slim cocones are stable under right whiskering.

Slim cocones are in a sense absolute left lifting diagrams which have a leg pointing the wrong way.

We say that κ is **colimiting** if it is invertible.

Proposition 6.2. *If $\kappa : \alpha u \Rightarrow x$ is a slim colimiting cocone, then (u, κ^{-1}) is an absolute left lift: $u = llift_\alpha x$ with an invertible cell.*

Corollary 6.3. *Slim colimiting cocones are unique up to isomorphism.*

Corollary 6.4. *A is continuous iff every parametrized object $a : X \to A$ is the vertex of a slim colimit cocone $\alpha u \to a$.*

Proof. ($\Rightarrow$) Since A is continuous we have a furthest left adjoint $\lambda \dashv \alpha$ which is always an abolute left lift $llift_\alpha id_A$, moreover since it is a coreflection the two cell is invertible. Thus precomposing with any parametrized object $a : X \to A$ gives an absolute left lift with invertible two cell $\lambda a = llift_\alpha a$.

($\Leftarrow$) Take the slim colimit cocone associated to $id_A : A \to A$ to obtain an absolute $llift_\alpha id_A$ together with an invertible two cell, thus this left lift will be the desired left adjoint. ☐

Definition 6.5. Let $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ be lax-idempotent, and let (X, α) be an algebra.

A **way-below arrow** between generalized elements $x, y : A \to X$ is a 2-cell $\gamma : x \to y$ such that the 2-cell obtained from the counit of the adjunction defining the algebra structure $\alpha\eta_X x \to y$ is a slim cocone.

Proposition 6.6. *A parametrized object $x : A \to X$ is compact iff the identity $x \to x$ is a way-below arrow*

7 The counterexample

Definition 7.1. A subset $C \subseteq D$ of a dcpo D is **Scott-closed** if it is down-closed and closed under directed joins.

Lemma 7.2. *(Cf. Lemma 4 [VT04]) There is a left adjoint* $\mathsf{cl} : \mathsf{Dcpo} \to \mathsf{Sup}$ *making the forgetful functor* $U : \mathsf{Sup} \to \mathsf{Dcpo}$ *monadic. It sends a dcpo D to its lattice* $\mathsf{cl}(D)$ *of Scott-closed sets. Moreover the induced monad* $\mathsf{cl} := U \circ \mathsf{cl}$ *is lax-idempotent.*

Remark 7.3. The lax idempotence can be seen from the framework of conservative cocompletions [Lam]. Since $X \in \mathsf{Dcpo}$, $\mathsf{cl}(X)$ adds in the *missing* finite joints while keeping the directed joins.

The following proposition tells us that we indeed recover the right notion of compactness for cl from [HZ09], which they called *C-compactness*.

Proposition 7.4. *An element $x : \mathbf{1} \to L$ of a suplattice is* cl***-compact*** *if and only if*

$$c \leqslant \bigvee A \Rightarrow c \in A$$

for all $A \in \mathsf{cl}(L)$.

Proof. By definition x is cl-compact iff $\downarrow x = \mathsf{lift}_x \bigvee$, that is

$$\downarrow x \subset_c lA \Leftrightarrow x \leqslant \bigvee A.$$

But the left to right direction is trivial giving us the desired result. ☐

Proposition 7.5. *Let D be a dcpo, and $\mathcal{A} \subseteq_{\mathsf{cl}} \mathsf{cl}(D)$ a Scott-closed set of Scott closed sets. Then $\bigcup \mathcal{A}$ is Scott-closed in D.*

Proof. Consider a directed family $a : (I, \leqslant) \to \bigcup \mathcal{A}$. Then for every $i \in I$ there exists an $A_i \in \mathcal{A}$ with $a_i \in A_i$. Define $A'_i = \bigcap_{j \geqslant i} A_j$. Then the A'_i are also all closed and in $\mathcal{A}$ and contain a_i. We have $a = \bigvee_i a_i \in \bigcup_i A'_i \subseteq \bigvee_i A'_i \in \mathcal{A}$ and therefore $a \in \bigcup \mathcal{A}$. □

Remark 7.6. The equalizer (inverter)

$$\mathsf{K}(\mathrm{T}D) \to \mathrm{T}D \rightrightarrows \mathrm{T}^2 D$$

consists precisely of the cl-compact elements of $\mathrm{T}D$. It clearly contains all principal downsets, but there there might be more!
Consider Johnstone's dcpo $\mathcal{J}$ [Joh06] whose underlying set is $\mathbb{N} \times \mathbb{N}_\infty$ and where $(i, n) \leqslant (j, m)$ iff either $i = j$ and $n \leqslant m$ or $m = \infty$ and $n \leqslant j$.

Proposition 7.7. *$\mathcal{J}$ is cl-compact in $\mathsf{cl}(\mathcal{J})$.*

Proof. Assume that $\mathcal{J}$ is the supremum (and therefore the union) of a closed set $\mathcal{A} \subseteq \mathsf{cl}(\mathcal{J})$. Then for all i there is an $A_i \in \mathcal{A}$ such that $(i, \infty) \in A_i$ and therefore $B_i = \mathbb{N} \times \{0, 1, \ldots, i\} \in \mathcal{A}$ since it's a closed subset of A_i. Then J is the directed join of the B_i and it must be contained in $\mathcal{A}$ since since $\mathcal{A}$ is assumed to be closed. □

We observe that $\mathrm{T}\mathcal{J}$ is $\mathcal{J}$ with a new greatest element adjoined. But in $\mathsf{K}\mathrm{T}\mathcal{J}$, the original $\mathcal{J}$ is again closed and cl-compact and not principal, so $(\mathsf{K}\mathrm{T})^2\mathcal{J}$ adds yet another element *under* the greatest element added in the first step. It seems that transfinite iteration never stops.

Corollary 7.8. *The induced adjunction* $\mathsf{Free} \dashv \mathsf{K}$ *from theorem 5.15 need not be idempotent. That is there are non pre-mnemetic lax-idempotent monads.*

References

[ADL23] Nathanael Arkor, Ivan Di Liberti, and Fosco Loregian. "Adjoint functor theorems for lax-idempotent pseudomonads". In: *arXiv* (June 2023). DOI: `10.48550/arXiv.2306.10389`. eprint: `2306.10389`.

[AR01] Jiří Adámek and Jiří Rosický. "On sifted colimits and generalized varieties." In: *Theory and Applications of Categories [electronic only]* 8 (2001), pp. 33–53. ISSN: 1201-561X. URL: `https://eudml.org/doc/121308`.

[Awo06] Steve Awodey. *Category Theory.* Oxford, England: Oxford University Press, 2006.

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. "Two-dimensional monad theory". In: *J. Pure Appl. Algebra* 59.1 (July 1989), pp. 1–41. ISSN: 0022-4049. DOI: `10.1016/0022-4049(89)90160-6`.

[Blo25] Thomas Blom. "Lax idempotent monads in homotopy theory". Talk presented at Category Theory 2025 (CT2025), Max Planck Institute for Mathematics, Bonn. July 2025. URL: `https://conference.math.muni.cz/ct2025/data/uploads/slides/blom.pdf`.

[CJ95] Aurelio Carboni and Peter Johnstone. "Connected limits, familial representability and Artin glueing". In: *Math. Struct. Comput. Sci.* 5.4 (Dec. 1995), pp. 441–459. ISSN: 1469-8072. DOI: `10.1017/S0960129500001183`.

[CS09] G. S. H. Cruttwell and Michael A. Shulman. "A unified framework for generalized multicategories". In: *arXiv* (July 2009). DOI: `10.48550/arXiv.0907.2460`. eprint: `0907.2460`.

[CV98] A. Carboni and E.M. Vitale. "Regular and exact completions". In: *Journal of Pure and Applied Algebra* 125.1 (1998), pp. 79–116. DOI: `https://doi.org/10.1016/S0022-4049(96)00115-6`.

[Di +17] Ivan Di Liberti et al. *Formal Category Theory: A Course Held at Masaryk University*. Lecture notes from a reading seminar at Masaryk University. 2017. URL: `https://ncatlab.org/nlab/files/DLHLL-FormalCategoryTheory.pdf`.

[DL18] Ivan Di Liberti and Fosco Loregian. "Accessibility and presentability in 2-categories". In: *arXiv* (Apr. 2018). DOI: `10.48550/arXiv.1804.08710`. eprint: `1804.08710`.

[EM07] A. D. Elmendorf and M. A. Mandell. "Permutative categories, multicategories, and algebraic K-theory". In: *arXiv* (Sept. 2007). DOI: `10.2140/agt.2009.9.2391`. eprint: `0710.0082`.

[GG24] Mai Gehrke and Sam van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*. Cambridge, England, UK: Cambridge University Press, Feb. 2024. ISBN: 978-1-00934968-0. DOI: `10.1017/9781009349680`.

[Gie+80] Gerhard Gierz et al. *A Compendium of Continuous Lattices*. en. Berlin, Heidelberg: Springer, 1980. ISBN: 978-3-642-67678-9. DOI: `10.1007/978-3-642-67678-9`. URL: `http://link.springer.com/10.1007/978-3-642-67678-9` (visited on 04/22/2025).

[Her01] Claudio Hermida. "From coherent structures to universal properties". In: *J. Pure Appl. Algebra* 165.1 (Nov. 2001), pp. 7–61. ISSN: 0022-4049. DOI: `10.1016/S0022-4049(01)00008-1`.

[HZ09] Weng Kin Ho and Dong Sheng Zhao. "Lattices of Scott-closed sets". In: *Comment. Math. Univ. Carolin.* 50.2 (2009), pp. 297–314. ISSN: 0010-2628. URL: `https://eudml.org/doc/32500`.

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, Volume 1*. Oxford, England: Clarendon Press, 2002.

[Joh06] Peter T. Johnstone. "Scott is not always sober". In: *Continuous Lattices*. Berlin, Germany, Oct. 2006, pp. 282–283. DOI: `10.1007/BFb0089911`.

[Joh82] Peter T. Johnstone. *Stone Spaces*. Cambridge, England, UK: Cambridge University Press, 1982. ISBN: 978-0-52133779-3. URL: `https://books.google.fr/books/about/Stone_Spaces.html?id=CiWwoLNbpykC&redir_esc=y`.

[KL97] G. M. Kelly and Stephen Lack. "On property-like structures." In: *Theory and Applications of Categories [electronic only]* 3 (1997), pp. 213–250. ISSN: 1201-561X. URL: `https://eudml.org/doc/120206`.

[Koc13] Anders Kock. "Fibrations as Eilenberg-Moore algebras". In: *arXiv* (Dec. 2013). DOI: `10.48550/arXiv.1312.1608`.

[Koc95] Anders Kock. "Monads for which structures are adjoint to units". In: *J. Pure Appl. Algebra* 104.1 (Oct. 1995), pp. 41–59.

[KS06] G. M. Kelly and Ross Street. "Review of the elements of 2-categories". In: *Category Seminar*. Berlin, Germany: Springer, Aug. 2006, pp. 75–103. ISBN: 978-3-540-37270-7. DOI: 10.1007/BFb0063101.

[Lac00] Stephen Lack. "A Coherent Approach to Pseudomonads". In: *Adv. Math.* 152.2 (June 2000), pp. 179–202. ISSN: 0001-8708. DOI: 10.1006/aima.1999.1881.

[Lam] Joachim Lambek. *Completions of Categories*. Berlin, Germany: Springer. ISBN: 978-3-540-34840-5. URL: https://link.springer.com/book/10.1007/BFb0077265.

[Lei98] Tom Leinster. "Basic Bicategories". In: *arXiv* (Oct. 1998). DOI: 10.48550/arXiv.math/9810017. eprint: math/9810017.

[Mes06] Bachuki Mesablishvili. "Monads of effective descent type and comonadicity." In: *Theory and Applications of Categories [electronic only]* 16 (2006), pp. 1–45. ISSN: 1201-561X. URL: https://eudml.org/doc/125870.

[MRW12] Francisco Marmolejo, Robert Rosebrugh, and R. J. Wood. "Completely and totally distributive categories I". In: *J. Pure Appl. Algebra* 216.8 (Aug. 2012), pp. 1775–1790. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2012.02.017.

[MW12] F. Marmolejo and R. J. Wood. "Kan extensions and lax idempotent pseudomonads". In: *Theory Appl. Categ.* 26 (May 2012). ISSN: 1201-561X. URL: https://www.researchgate.net/publication/265572263_Kan_extensions_and_lax_idempotent_pseudomonads.

[PH21] Dusko Pavlovic and Dominic Hughes. "The nucleus of an adjunction and the Street monad on monads". In: *Theory Appl. Categ.* (Jan. 2021). URL: https://par.nsf.gov/biblio/10312054-nucleus-adjunction-street-monad-monads.

[PT21] Paolo Perrone and Walter Tholen. "Kan extensions are partial colimits". In: *arXiv* (Jan. 2021). DOI: 10.1007/s10485-021-09671-9. eprint: 2101.04531.

[RV01] J. Rosický and E. M. Vitale. "Exact completion and representations in abelian categories." In: *Homology, Homotopy and Applications* 3.3 (2001), pp. 453–466. ISSN: 1532-0073. URL: https://eudml.org/doc/49367.

[SW78] Ross Street and Robert Walters. "Yoneda structures on 2-categories". In: *J. Algebra* 50.2 (Feb. 1978), pp. 350–379. ISSN: 0021-8693. DOI: 10.1016/0021-8693(78)90160-6.

[Tow22] Christopher Townsend. "Stably locally compact locales are dual to continuous posets". In: *Journal of Pure and Applied Algebra* 226.2 (2022), p. 106836. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2021.106836. URL: https://www.sciencedirect.com/science/article/pii/S0022404921001778.

[VT04] S.J. Vickers and C.F. Townsend. "A universal characterization of the double powerlocale". In: *Theoretical Computer Science* 316.1 (2004). Recent Developments in Domain Theory: A collection of papers in honour of Dana S. Scott, pp. 297–321. ISSN: 0304-3975. DOI: https://doi.org/10.1016/j.tcs.2004.01.034. URL: https://www.sciencedirect.com/science/article/pii/S0304397504000878.

[Wal17] Charles Walker. "Yoneda structures and KZ doctrines". In: *arXiv* (Mar. 2017). DOI: 10.1016/j.jpaa.2017.07.004. eprint: 1703.08693.

[Zöb76] Volker Zöberlein. “Doctrines on 2-Categories.” In: *Mathematische Zeitschrift* 148 (1976), pp. 267–280. URL: http://eudml.org/doc/172368.

A Background on Pseudomonads

Pseudomonads encode structure which is defined up to isomorphism, for example a monoidal structure on a category. A complete set of references for the constructions in this section is [Lei98; BKP89; Lac00].

Definition A.1. Let $\mathcal{B}$ and $\mathcal{C}$ be 2-categories, a **pseudofunctor** $\mathsf{F} : \mathcal{B} \to \mathcal{C}$ consists of the following data:

- A function $\mathsf{F}_0 : \mathcal{B}_0 \to \mathcal{C}_0$, where for $A \in \mathcal{B}$ we denote $\mathsf{F}_0(A)$ as $\mathsf{F}(A)$
- For each pair of objects $A, B \in \mathcal{B}$, a functor $\mathsf{F}_{A,B} : \mathcal{B}(A, B) \to \mathcal{C}(A, B)$ where we again denote $\mathsf{F}_{A,B}(f)$ as $\mathsf{F}(f)$
- For each object $A \in \mathcal{B}$ an invertible 2-cell $\mathsf{F}_{\mathrm{id}_A} : \mathrm{id}_{\mathsf{F}(A)} \Rightarrow \mathsf{F}(\mathrm{id}_A)$
- For each triple $A, B, C \in \mathcal{B}$ a natural isomorphism (natural in $f : A \to B$ and $g : B \to C$) $\mathsf{F}_{g,f} : \mathsf{F}(g) \circ \mathsf{F}(f) \Rightarrow \mathsf{F}(g \circ f)$

This datum is required to make two diagrams commute which appear in [Lei98].

Definition A.2. Given 2-categories $\mathcal{B}$ and $\mathcal{C}$ with pseudofunctors $\mathsf{F}, \mathsf{G} : \mathcal{B} \to \mathcal{C}$ between them, a **pseudonatural transformation** $\alpha : \mathsf{F} \Rightarrow \mathsf{G}$ is given by:

- For each object $A \in \mathcal{B}$, an arrow $\alpha_A : \mathsf{F}(A) \to \mathsf{G}(A)$ in $\mathcal{C}$
- For each arrow $f : A \to B \in \mathcal{B}$, an invertible 2-cell $\alpha_f : \mathsf{G}(f) \circ \alpha_A \Rightarrow \alpha_B \circ \mathsf{F}(f)$.

And this datum has to satisfy coherence diagrams [Lei98].

Definition A.3. A **modification** between pseudonatural transformations $\alpha, \beta : \mathsf{F} \Rightarrow \mathsf{G}$, denoted $\theta : \alpha \Rrightarrow \beta$ is given by a family of 2-cells $\theta_A : \alpha_A \Rightarrow \beta_A$ indexed by objects $A \in \mathcal{B}$ such that for each arrow $f : A \to B$ the following square commutes:

$$\begin{array}{ccc} \mathsf{G}(f)\alpha_A & \xrightarrow{G(f)\theta_A} & \mathsf{G}(f)\beta_A \\ {\scriptstyle \alpha_f}\downarrow & & \downarrow{\scriptstyle \beta_f} \\ \alpha_B\mathsf{F}(f) & \xrightarrow{\theta_B F(f)} & \beta_B\mathsf{F}(f) \end{array} .$$

Definition A.4. Let $\mathcal{B}$ be a 2-category, a **pseudomonad** $(\mathrm{T} : \mathcal{B} \to \mathcal{B}, \mu, \eta)$ consists of the following data:

- A pseudofunctor $\mathrm{T} : \mathcal{B} \to \mathcal{B}$
- pseudonatural transformations: $\mu : \mathrm{T}^2 \Rightarrow \mathrm{T}$, $\eta : \mathrm{id} \Rightarrow \mathrm{T}$
- invertible modifications: $\lambda : \mu \circ \eta\mathrm{T} \Rrightarrow \mathrm{id}$, $\rho : \mu \circ \mathrm{T}\eta \Rrightarrow \mathrm{id}$ and $\alpha : \mu \circ \mathrm{T}\mu \Rrightarrow \mu \circ \mu\mathrm{T}$.

These are required to satisfy two diagrams which can be found in [Lac00].

B Noetherian objects

With the greatest amount of bad intent one could call an algebra (X, α) of a lax-idempotent monad T on a 2-category **Noetherian** if $\mathrm{id}_X : X \to X$ is compact in the sense of Section 4.

It is not too hard to show that these are the fixed points of the monad.

Lemma B.1. *An algebra (X, α) is Noetherian if and only if $\alpha : \mathrm{T}X \to X$ is an equivalence.*

Proof. The only if direction is clear. For the other direction, take (X, α) Noetherian, so id_X is compact, thus $\eta_X = \eta_X \mathrm{id}_X = \mathsf{lift}_\alpha \mathrm{id}_X$ is an absolute left lift. Thus by Section 2.2

$$\eta_X \dashv \alpha.$$

Thus η_X is an algebra morphism by Section 3.3. Hence by the universal property of free algebras we have that

$$\eta_X \alpha \eta_X \cong \eta_X,$$

which implies that $\eta_X \alpha \cong \mathrm{id}_{\mathrm{T}X}$. Thus α is an equivalence. □

Proposition B.2. *Any reflective subalgebra of a Noetherian algebra (X, α) is Noetherian.*

Proof. Take a reflective subalgebra $A \subseteq X$ with inclusion reflection $r \dashv i$. By Section 4, r preserves compact objects, thus by absoluteness of compact objects $r \circ \mathrm{id}_X \circ i$ is compact in A. Hence id_A is compact in A, so A is Noetherian. □

Proposition B.3. *Any algebra coreflection of a Noetherian algebra (X, α) is Noetherian.*

Proof. Same proof. □

Corollary B.4. *Noetherian spaces are closed under quotients and subspaces.*

C Definable compactness

We can do a tiny bit better than 5.15 even if we don't have inverters.

Definition C.1. Compactness is **definable** for an algebra (X, α) if there is a universal compact arrow $\iota : \mathsf{K}X \to X$, that is:

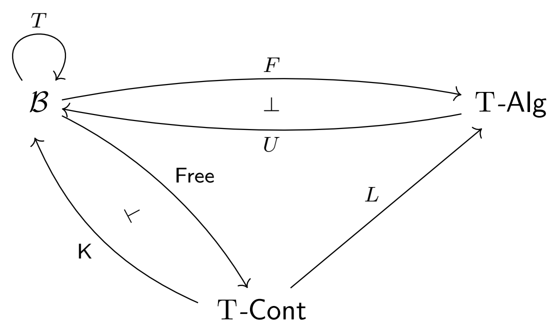
- ι_X is compact
- For every object $Y \in \mathcal{B}$: $- \circ \iota : \mathcal{B}(Y, X) \to \mathcal{B}(Y, A)$ is full and faithful with its replete image in compact arrows for X

This defines the 2-category $\mathrm{T}\text{-}\mathsf{Alg}_{\mathsf{def}}$ of algebras with definable compactness, coherent algebra morphisms and 2-cells of $\mathcal{B}$.

Proposition C.2. *Compactness is definable for continuous algebras precisely when the inverter of their canonical 2-cell exists.*

With this definition in hand we can drop the restrictions of theorem 5.15.

Theorem C.3. *Any lax-idempotent monad* $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ *induces an adjunction between* $\mathcal{B}$ *and* $\mathrm{T}\text{-}\mathsf{Alg}_{\mathsf{def}}$ *such that* $L \circ \mathsf{Free} \cong F$*:*



Definition C.4. An algebra (X, α) with definable compactness 'has enough compact objects' if the counit $c_X : \mathsf{TK}X \to X$ is an equivalence.

Corollary C.5. *A lax-idempotent monad is mnemetic precisely is* Free *is a local equivalence and pre-mnemetic precisely if* $\mathsf{Free} \dashv \mathsf{K}$ *is idempotent.*

We now quickly return to the proof of theorem 5.32:

Proposition C.6. *Assume* $\mathrm{T} : \mathcal{B} \to \mathcal{B}$ *is mnemetic and* $\mathcal{B}$ *has inverters of adjoint cylinders. And assume* $A \in \mathcal{B}$ *and that* (X, α, λ_X) *is a continuous algebra such that*

$$X \underset{l}{\overset{i}{\rightleftarrows}} \mathrm{T}A \quad (l \dashv i)$$

with $i : X \to \mathrm{T}A$ *a representably full and faithful continuous algebra morphism. In other words* X *is a reflective continuous subalgebra of* $\mathrm{T}A$*. Then* X *has enough compact objects, that is* $c_X : \mathsf{TK}X \to X$ *is an equivalence of continuous algebras.*

Proof. We construct a pseudoinverse to c_X. Denote $\iota_X : \mathsf{K}X \to X$ the canonical inclusion and notice that by 5.5 η_A is compact. By Section 4 l preserves compact objects, thus $l\eta_A : A \to X$ is compact. The universal property of ι_X gives us a factorizing morphism $f : A \to \mathsf{K}X$ such that $\iota_X f \cong l\eta_A$

$$\begin{array}{ccc} X & \xleftarrow{\;l\;} & \mathrm{T}A \\ {\scriptstyle \iota_X}\uparrow & & \uparrow{\scriptstyle \eta_A} \\ \mathsf{K}X & \xleftarrow[f]{\;\cdots\;} & A \end{array} .$$

Let $g : X \to \mathsf{TK}X$ be the composite

$$X \xrightarrow{\;i\;} \mathrm{T}A \xrightarrow{\;\mathrm{T}f\;} \mathsf{TK}X .$$

Since i is a continuous algebra morphism, g is continuous algebra morphism.

So $g \circ c_X : \mathsf{TK}X \to \mathsf{TK}X$ is a continuous algebra morphism defined on free algebras, thus by mnemeticity $\eta_{\mathsf{K}X}\mathsf{K}(g \circ c_X) \cong g \circ c_X \circ \eta_{\mathsf{K}X}$. Now since we are working with algebra morphisms defined over free algebras we can check equivalences on generators

$$\begin{aligned} g \circ c_X \eta_{\mathsf{K}X} &= \mathrm{T}(f) i \alpha \mathrm{T}(\iota_X) \eta_{\mathsf{K}X} \\ &\cong \mathrm{T}(f) i \alpha \eta_X \iota_X \\ &\cong \mathrm{T}(f) i \iota_X . \end{aligned}$$

Thus $\mathrm{T}(f) i \iota_X \cong \eta_{\mathsf{K}X} \mathsf{K}(g \circ c_X)$. So precomposing with c_X gives us on one hand

$$c_X \eta_{\mathsf{K}X} \mathsf{K}(g \circ c_X) \cong \iota_X \mathsf{K}(g \circ c_X).$$

And on the other hand

$$\begin{aligned} c_X \mathrm{T}(f) i \iota_X &= \alpha \mathrm{T}(\iota_X) \mathrm{T}(f) i \iota_X \\ &\cong \alpha \mathrm{T}(\iota_X f) i \iota_X \\ &\cong \alpha \mathrm{T}(l \eta_A) i \iota_X \\ &\cong \alpha \mathrm{T}(l) \mathrm{T}(\eta_A) i \iota_X \\ &\cong l \mu_A \mathrm{T}(\eta_A) i \iota_X \\ &\cong l i \iota_X \\ &\cong \iota_X \end{aligned}$$

Hence $\mathsf{K}(g \circ c_X) \cong \mathrm{id}_{\mathsf{K}X}$ and by theorem 5.24

$$(g \circ c_X) \cong \mathsf{Free}(\mathrm{id}_{\mathsf{K}X}).$$

And the other direction is now immediate

$$\begin{aligned} c_X g &= \alpha \mathrm{T}(\iota_X) \mathrm{T}(f) i \\ &\cong \alpha \mathrm{T}(\iota_X f) i \\ &\cong \alpha \mathrm{T}(l \eta_A) i \\ &\cong \alpha \mathrm{T}(l) \mathrm{T}(\eta_A) i \\ &\cong l \mu_A \mathrm{T}(\eta_A) i \\ &\cong l i \\ &\cong \mathrm{id}_X \end{aligned}$$

So c_X is an equivalence of continuous algebras. □

Remark C.7. A similar argument shows that assuming T is lax idempotent and $X \to \mathrm{T}A$ is a reflective subalgebra, then c_X has a section. The problem is that c_X need not be representably fully faithful in general as in Section 7.

D A big list of examples

In this final section we wanted to show that lax-idempotent monads come in many forms each giving rise to different interesting notions of generator!

The shape column indicates the kind of diagrams that we cocomplete under, all means all

(small) diagrams, finite means only finite diagrams, etc. The case of cocartesian arrows is in reference to turning an arbitrary functor $F : E \to B$ over a fixed base B into an opfibration $\mathsf{OP}(F) : F \downarrow B \to B$. This defines the action of a lax-idempotent monad [Koc13].
Question marks correspond to notions of compactness we don't have a name for yet. $\mathcal{D}$ and $\mathcal{D}_{fg}$ are the down-set constriction and the finitely generated down-set construction respectively. Fam is the families construction/ free coproduct completion. Reg is the reg/lex completion, Ex is the ex/lex completion. And finally Env is the envelope of a multicategory, which has been shown to be lax-idempotent in for example [Her01].

2-Category	**Shape**	**L.I. Monad**	**Algebras**	T-**Compactness**	**Reference**
Pos	all	$\mathcal{D}$	Sup	Completely-join-prime element	[GG24]
Pos	directed	Idl	Dcpo	join-prime element	[Gie+80]
Pos	finite	$\mathcal{D}_{fg}$	$\vee$-Lat	finite/compact element	[GG24] (Exercise 1.3.8)
CAT	all	$\mathcal{P}$	CoCompCat	atomic object	[PT21]
CAT	directed	Ind	IndCoCompCat	finitely presented object	[Joh82] (Ch. VI)
CAT	discrete	Fam	CoprodCocompCat	connected object	[CJ95]
CAT	sifted	SInd	SIndCoCompCat	?	[AR01]
Lex	image factorizations	Reg	RegCat	regular projective object	[CV98]
Lex	effective quotients	Ex	ExCat	effective projective object	[CV98]
AddCat	effective quotients	Ex	AbCat	effective projective object	[RV01]
CAT/B	cocartesian arrows	OP	OpFib(B)	?	[Koc13]
MultiCat	representables	Env	MonCat	?	[Her01]

This last example Env : MultiCat $\to$ MultiCat is special in another way. Namely it is an example of theorem 5.30, which has been shown in [EM07] for symmetric Multicategories.