Compactness for Lax Idempotent Monads

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- An abstract framework for studying compactness for Lax-Idempotent monads
- An application to opfibrations

Compactness for poset-enriched categories

Familiar properties:

- In a Poset admitting all directed joins (DCPO) D, $x \in D$ is called compact if for any directed S, we have:

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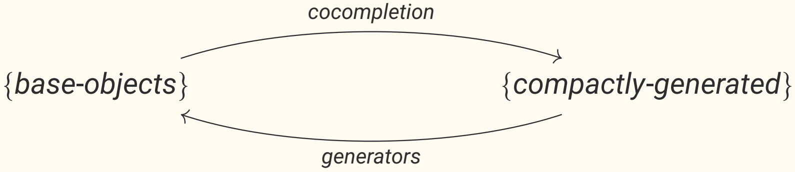
With related theorems:

- Pos is equivalent to the category of algebraic DCPO's with DCPO morphisms preserving compact elements
- Pos is equivalent to the category of Sup-Lattices generated by their completely join prime elements with sup lattice morphisms preserving these.

Motivation

We want to understand equivalences of the shape:



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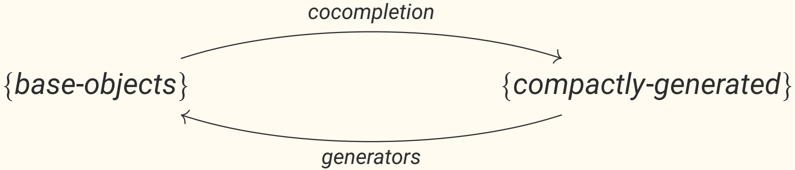
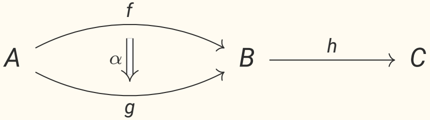


This is for example the shape of Gabriel-Ulmer Duality

Preliminaries: 2-Categories

Intuition. A 2-category is a category-like structure with objects, arrows and 2-cells which are arrows between arrows.



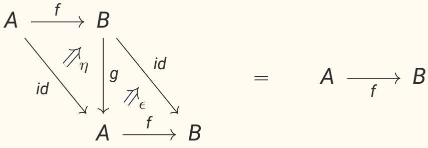
Examples:

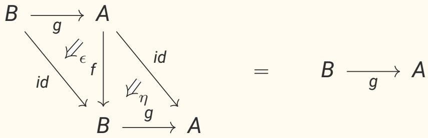
Pos: Posets, order preserving functions, pointwise comparisons

Cat: Locally small categories, functors, natural transformations

Preliminaries: Adjunctions in 2-Categories

Definition. An adjunction $f \dashv g$ in a 2-category is a pair of arrows $f : a \to b$, $g : b \to a$ with 2-cells $\eta : 1 \Rightarrow gf$ and $\epsilon : fg \Rightarrow 1$ satisfying the triangle identities:





Basic Notions of Lax-Idempotent Monads

Lax-idempotent monads give us a setting for "cocompletions":

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3. for multicategories: freely making them representable
4. for hyperdoctrines: freely adding existential quantification between the fibers

Definition.[1] A (pseudo) 2-monad on a 2-category $\mathcal{K}$, consists of the following data:

[1] Blackwell, R., Kelly, G. M., & Power, A. J. (1989). **Two-dimensional monad theory.** *J. Pure Appl. Algebra*, *59*(1), 1–41. https://doi.org/10.1016/0022-4049(89)90160-6

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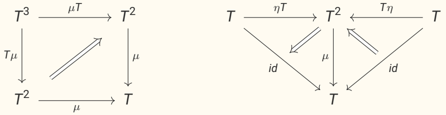
- pseudo-natural transformations $\mu : T^2 \to T$ and $\eta : 1 \to T$
- invertible modifications:



Satisfying coherence axioms

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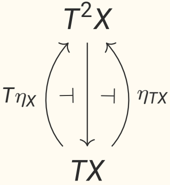

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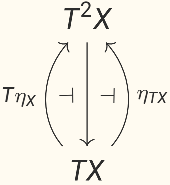

This adjoint cylinder induces a 2-cell $\theta : T\eta \Rightarrow \eta T$, so the idempotence becomes lax

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Non-example. Free monoidal category on a category

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$\mathcal{D}$: Pos $\rightarrow$ Pos

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$\mathcal{D} : \mathsf{Pos} \to \mathsf{Pos}$

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$\mathsf{Idl} : \mathsf{Pos} \to \mathsf{Pos}$

ideals, μ: directed join, η: principal ideal

Algebras of Lax-Idempotent Monads

Proposition.[2] The (pseudo-) algebras of a lax-idempotent monad $(T : B \to B, \mu, \eta)$ are pairs $(X, \alpha : TX \to X)$ such that $\alpha \dashv \eta_X$ and $\alpha\eta_X \cong id$.

- For $\mathcal{P}$: cocomplete categories

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- For $\mathcal{D}$: complete join-semi lattices (Sup-Lattices)

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For l.i. monads being an algebra is a property!

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Forms of compactness

For a Sup-Lattice X, $x \in X$ is completely join prime if:

$$x \leqslant \bigvee S \Leftrightarrow \exists s \in S : x \leqslant s$$

When S is downwards closed we can restate this as:

$$x \leqslant \bigvee S \Leftrightarrow \downarrow x \subseteq S$$

So the "compact" elements with respect to the down-set monad are the ones where the unit η_X behaves like a left adjoint to the algebra α.

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$$x \leqslant \alpha S \Leftrightarrow \eta_X x \subseteq S$$

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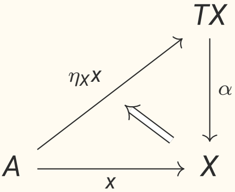
General compactness

Definition. A morphism $x : A \to X$ for an algebra (X, α) is T-compact if we have that:

$$\frac{\eta_X x \Rightarrow u}{x \Rightarrow \alpha u}$$

natural in A. Formally this means that:



is an absolute left lifting diagram, where the 2-cell is part of the iso $\alpha\eta_X \cong id$.

General compactness on Categories

- For $\mathfrak{X}$ cocomplete, $x : \mathbf{1} \to \mathfrak{X} : x$ is $\mathcal{P}$-compact iff it is atomic in the sense that
$$\mathrm{hom}(x, -) : \mathfrak{X} \to \mathrm{SET}$$
preserves small colimits.
- For $\mathfrak{X}$ ind-cocomplete, $x : \mathbf{1} \to \mathfrak{X}$ is Ind-compact when it is a compact-object, i.e.
$$\mathrm{hom}(x, -) : \mathfrak{X} \to \mathrm{SET}$$
preserves all filtered colimits.

General compactness on Posets

- For a DCPO D, $x : \mathbf{1} \to D$ is Idl-compact when it is a compact-element, i.e.

$$x \leqslant (-) : D \to 2$$

  preserves directed joins
- For a Sup-Lattice L, $x : \mathbf{1} \to L$ is $\mathcal{D}$-compact when it is a completely join prime-element, i.e.

$$x \leqslant (-) : L \to 2$$

  preserves all joins

Opfibration Monad

Definition. For $F : E \to B$, we define the comma category $F \downarrow B$ with objects $f : Fe \to b$ and morphisms commuting squares.

$$\begin{array}{ccc} e & \qquad & Fe & \xrightarrow{\;f\;} & b \\ {\scriptstyle g}\downarrow & & {\scriptstyle Fg}\downarrow & & \downarrow{\scriptstyle h} \\ e' & & Fe' & \xrightarrow{\;f'\;} & b' \end{array}$$

Opfibration Monad

Proposition.[3] There is a lax-idempotent monad OP : $Cat/B \to Cat/B$ defined on objects by taking:

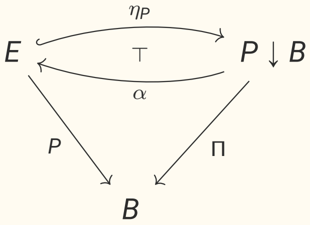
$$(F : E \to B) \mapsto (F \downarrow B \to B)$$

The unit $\eta_F : E \to F \downarrow B$ takes $e \mapsto id_{F(e)} : F(e) \to F(e)$
The multiplication μ acts via composition.

[3] Kock, A. (2013).**Fibrations as Eilenberg-Moore algebras.** *arXiv*. https://doi.org/10.48550/arXiv.1312.1608

Opfibration Monad

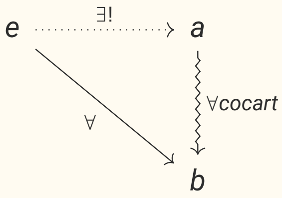
A $p : E \to B$ is an OP-algebra if $\eta_P : E \to P \downarrow B$ has a left adjoint which commutes with the projections.



Thus each $f : P(e) \to b$ has a specified cocartesian lift $e \to \alpha(f)$ given by the unit of the adjunction $\alpha \dashv \eta_P$.

General compactness for opfibrations

For a (split) opfibration $p : E \to B$, $e \in E$ is OP-compact if we have the following lifting property against cocartesian arrows:



In other words, e is OP-compact iff it is left orthogonal to cocartesian arrows in E.

Enough compact objects

Definition. An opfibration $p : E \to B$ has enough OP-compact objects when every $e \in E$ is the codomain of a cocartesian arrow with compact domain.

Proposition. An opfibration $p : E \to B$ with enough OP-compact objects is free, i.e. of the form $\mathrm{OP}(f)$ for some $f : C \to B$.

Mnemetic Monads

Proposition. For $T = \mathsf{OP}, \mathsf{Idl}, \mathcal{D}$ we have that $\mathsf{K}TX \simeq X$, where $\mathsf{K}(X,\alpha)$ is the universal compact arrow relative to the algebra (X,α).

Non-example. For the $\mathcal{P}$, $\mathsf{K}TX \not\simeq X$ in general. (Cauchy completion)

Definition. A lax-idempotent monad T is mnemetic when for any X, $\mathsf{K}TX \simeq X$.

Proposition. T is mnemetic iff the unit of the monad is the inverter of the 2-cell depicted:

$$X \xrightarrow{\eta_X} TX \underset{\eta_{TX}}{\overset{T\eta_X}{\rightrightarrows}} T^2X, \qquad \theta : T\eta_X \Rightarrow \eta_{TX}$$

Conclusions

What we saw:

- An abstract criterion for compactness
- A way of using it to extract theorems about free algebras

Ongoing work:

- Understand the lax idempotent monad on multicategories through this lens
- Way-Below arrows and Continuous Algebras